An Ocean of Possibilities

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Strangers on the Shore

For more than one hundred years we have paddled beside the great ocean of mathematical adventure. Between us we have taught millions of young people. A few have dived in and kept swimming, some have lingered on the shore playing in pools, but most have dipped their toes in and run like heck in the other direction never to return.

I would like to see them all - well, can't change the past, so all the ones in the future - lolling about joyfully in the depths. Why doesn't that happen? I think its because we aren't teaching students to swim - we're teaching them to play in rock pools.

Traditional mathematics teaching derives from the early Industrial Revolution. Compulsory, universal education was introduced to keep kids out of the mills - a worthy objective of course - but what were they to be taught while locked up in schools for several hours a day. Reading and writing ...and... arithmetic. Beyond school there was just a chance they would get a job in the burgeoning commercial arm of the industrial revolution - set up for life at their own sloping desk with candle, Dickensian eye-shade, dipping pen and giant ledger. With that vision the teaching style is consequent and appropriate. The children will have to learn to:

- give answers not ask questions
- follow recipes not explore possibilities
- be swift and accurate not contemplative and experimental

There is no need to learn to swim if you are only going to paddle in a rock pool.

But 'getting a job' is no longer either a necessary or sufficient reason for learning maths. It's probably at least fifty years since that argument has had relevance for students. Other equally irrelevant reasons from their point of view are 'you need it to pass an exam', or 'you might need this when you grow up' or 'because I said so'.

Although once relevant, a vision and pedagogy almost two centuries old is no longer appropriate. What can replace it? So many of us have tried for years and still there seems no change. I suspect that's because we too have been paddling in rock pools. They may be a little deeper, even closer to the breakers and a little more adventurous, but the tendency has been to use investigations because we feel they are great in their own right and so obviously much more than just practising skills. When asked, we still say we are teaching maths.

My replacement vision would be to stop teaching mathematics ... and start learning to work like a mathematician. After all, it is professional mathematicians who are out there swimming on a daily basis.

Learning to Work Like a Mathematician

Does anyone believe that a professional mathematician bounces into the office in the morning, turns on the computer and delights in doing all the exercises on the left hand side of the screen - perhaps checking the answers on the back of the monitor? No doubt many students, having experienced 10 years of compulsory mathematics education, suspect there is an element of truth in that image. Many who have been asked what they think a mathematician does (if they think they aren't all dead people from history) say 'sums'.

On the other hand a Year 1 teacher was impressed by the insight of a student whose response was 'They solve the world's hardest problems'.

Around 1993 the Mathematics Task Centre Project (HREF1) found out what professional mathematicians do by asking some. To a person their response was:

> *First give me an interesting problem.*

Now there's a guide for developing mathematics curriculum.

> *I have to teach ... (fractions) ... tomorrow. First I need to find a problem that involves fractions then think about the teaching craft that will interest my students in it.*

Note, a problem is not a variation on a previously learned situation requiring the application of a known procedure. Rather it is something that by its nature doesn't have an obvious response or perhaps even a way to begin. So, if the core of our curriculum is to become learning to work like a mathematician, we have to find a whole bunch of problems and tease out the teaching craft that fascinates, captivates and absorbs students in them.

We've been doing that for thirty years I hear you cry. True, but our raison d' être has not been mathematicians and how they work, but rather the problem itself and what a great learning experience it is. That's why the whole country did 'Frogs' ... and the students still didn't see the ocean behind them. The difference is in publishing, using the language of, making regular reference to, complimenting and celebrating students based on and evaluating against what it means to work like a mathematician.

Interviewing mathematicians resulted in the Working Mathematically document appended which is currently used by many schools as the core reference for their curriculum. In many ways the 'old' mathematics content now included in the syllabus doesn't really matter, but, whatever is chosen must contribute to kids learning to work like a mathematician as detailed by this document.

There is an ocean of possible 'interesting problems'. Let's take one and show why, if students are learning to work like a mathematician in happy, healthy, cheerful, productive, inspiring classrooms, it belongs in the curriculum.

Square Pairs

Beginning

Bob Vertes introduced me to this problem at an ATM conference around 2004. But he didn't tell me the problem - he *interested* me in the problem. His teaching craft was what caught my attention.

Bob handed out large number cards (1 - 18) to eighteen people, then continued...

> *You are guests at a party of 18 people. As you arrived the hostess has given you one of these number cards and told you to hold onto it until everyone comes. Now everyone is here and she says, "I want you to get to know each other. I want you all to find a partner so the sum of your numbers is a square number. It can be done."*

Bob knew what the problem was - pair the numbers from 1 to 18 to make squares - but he didn't start from that. He started by interesting me with a story context, physical involvement and a little mystery. Now he has introduced a challenge ... and I'm hooked.

Bob has satisfied the opening criteria for a mathematician's work:

> *First give me an interesting problem*

and before the students have solved it, I now pause my class for a moment to ask:

> *Do you have a problem?*
> *Are you interested in it?*

The response is obviously yes and that allows me to say:

> *Great, then you are working like a mathematician already because mathematicians say their work begins with an interesting problem.*

This comment is not just hot air. It is the starting point for Working Mathematically and at this point the teacher refers to the page which is in all the student journals and/or displayed beside the board in poster size. Interestingly, many teachers report that students no longer ask 'Why are we doin' this stuff sir?'

Middle

Back to the problem. Perhaps my students will find the correct pairing without too much trouble, but more often most have partners but some don't. Some pairs need to 'divorce' and reform. Now reasoning and working in community begin to replace the strategy of guess, check and work for yourself:

- 18 has to pair with 7 (to make 25) or with ... *Hey it can't go with anything else.*
- Are there any other must-pairs? If we do them first it might be easier.

Soon the solution will be found and recorded on the board, just like a mathematician would record in their journal.

Now I bring out my prepared sets of tiles numbered from 1 - 24. Alternatively I issue each pair of students with 3 pieces of scrap paper and ask them to fold and tear each into 8 pieces and number them from 1 to 24.

> *Just use the numbers 1 to 18 to start with and make sure you can pair up those party-goers. Then record an explanation of the problem and the solution in your journal.*

If I want my students to learn to work like a mathematician I have to take time to model how and what to record and provide opportunity to practise this craft.

Then the story continues - in fact, often students continue it by asking:

> *Could you pair up like this if there were a different number of people at the party sir?*

Perhaps the teaching craft decision to bring in hands-on materials and the time to process provided by the journal entry moment has rekindled interest and generated this question. The problem is moving towards becoming the students' and I can compliment them again for working like mathematicians because they have asked *What happens if...?* Of course they must now explore for themselves.

Divide your board into two parts. One for students to record solutions they find and the other to record insights such as *Add the highest and the second highest. The largest square must be equal to or less than this total.* There are only twelve possible party sizes to check - 2, 4, 6, ..., 20, 22, 24. Once they are found, encourage journaling again. Also, referring to Working Mathematically, ask *How have we worked like a mathematician so far?*

Can I suggest you stop reading here and work on the problem yourself for a while, then compare your own work to this scaffold.

A Working Mathematically curriculum has the simple objective - at all levels - of interesting students in learning to work like a mathematician, so we must keep drawing them back to this description to develop a 'can do' self-image.

End ... or is it another beginning

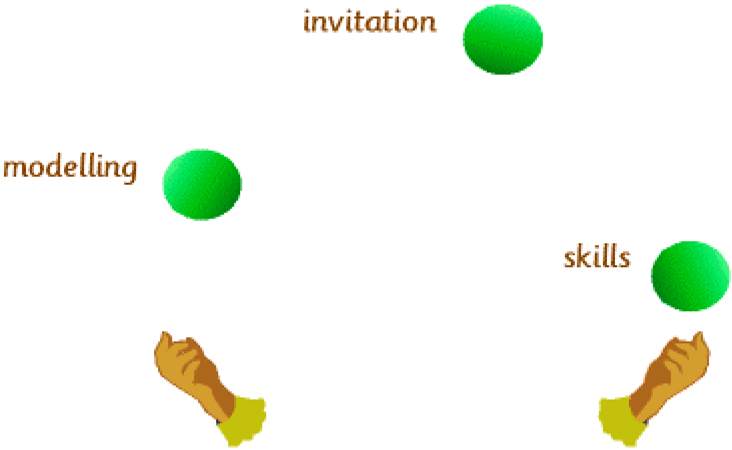
It isn't necessary for the continuing exploration of the problem, but it certainly is brilliant, that Bob learned about this problem from two modern day professional mathematicians. In their article *Partitions into Square-Pairs* (see below) Anderson & Walker state:

> *...there are aspects of this problem and its solution ... which could form the basis for an investigative project for pupils, involving experimentation (either on paper or through simple computer programs), conjecture and, more importantly, an opportunity to try to construct proofs.*

Kinda nice really. This stuff is not examinable and you would think that no one would give a tinkers about which number lists can square pair, yet this investigation is the work of professional mathematicians. In my replacement vision, the work of teachers like Bob Vertes is to interest kids in this problem and hundreds of others that are equally rich. Why? Because they are learning to work like mathematicians. Full stop, end of story, no other reason required.

Learning to Work Like...

If you are going to learn to work like anything there are three major components.



Someone has to show you how to do it. You have to have the opportunity to do it yourself and you will need to practise related skills.

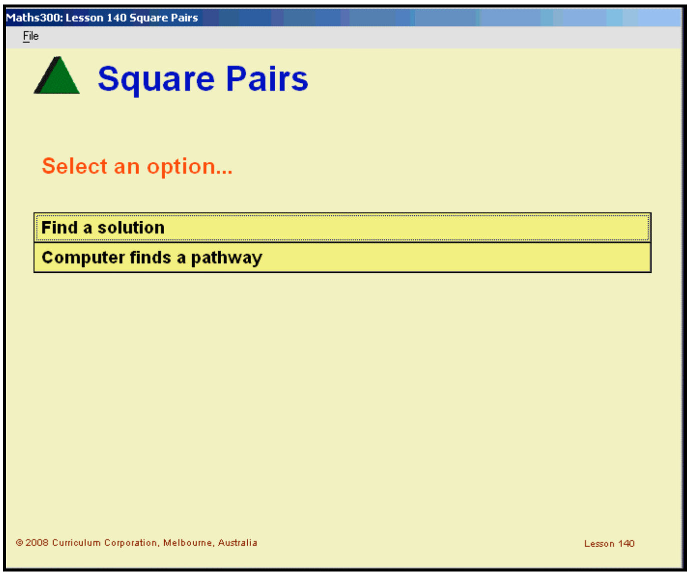
For example, learning to drive. At 16 (in Australia) a young person gets the learner's permit. Mum and dad could give them the car and the keys and leave them alone on the road for two years - an invitation to learn to drive. Or, mum and dad could say: *Mate, I have been doing this for 30 years. Sit here next to me for the next two years and I will show you how it's done.* - modelling how to drive. Or mum and dad could decide that the bit which trips up every other kid is the skill of reverse parking and spend two years practising (and no doubt testing) that.

Clearly none of these things alone will assist the learner driver to obtain their provisional licence. A successful learning to drive curriculum would involve juggling the time spent on each to find the appropriate balance for the student. So it is when the core curriculum is learning to work like a mathematician.

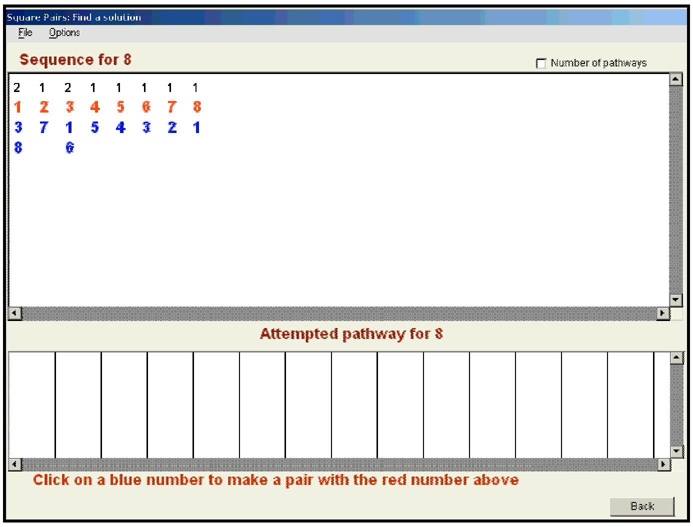
Square Pairs Continued

If you accepted my invitation above and tried the problem for yourself you will have found that some party sizes don't square pair. Anderson & Walker tell us there are only seven such lists. Students are challenged to find the seven and explain how they know that each one doesn't square pair. In explaining, students are encouraged to identify strategies chosen from the mathematician's strategy toolbox. Their explanation can be enhanced by using the software developed within

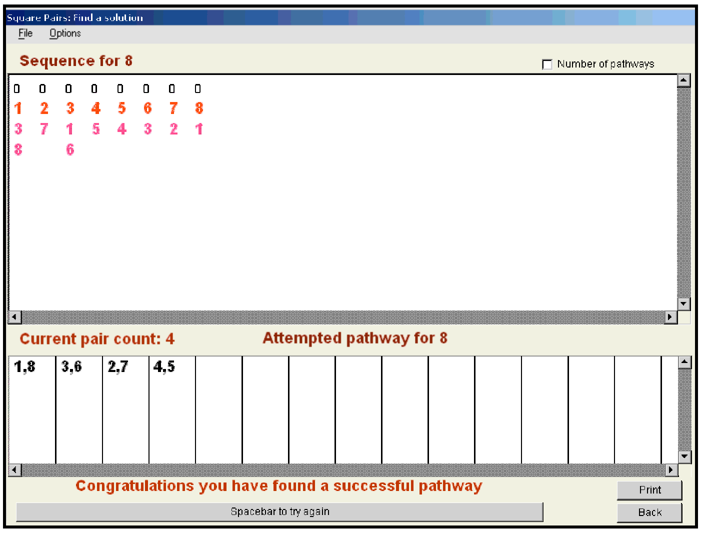
Maths300, HREF2, (or other if you have it) and an Interactive White Board. (Screen images shown are used with permission.)



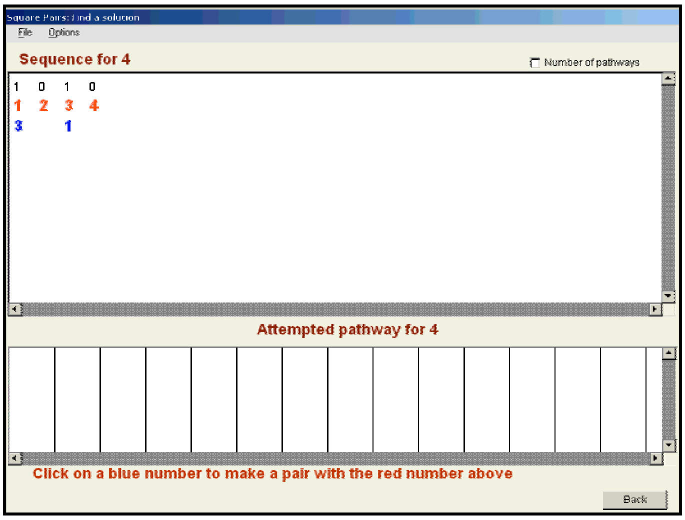
Note that the software is introduced some time after students have explored the problem by hand. To get a sense of how Option 1 works, let's examine a party size that does square pair. When the user chooses to test for a party of 8 people they are presented with this data:



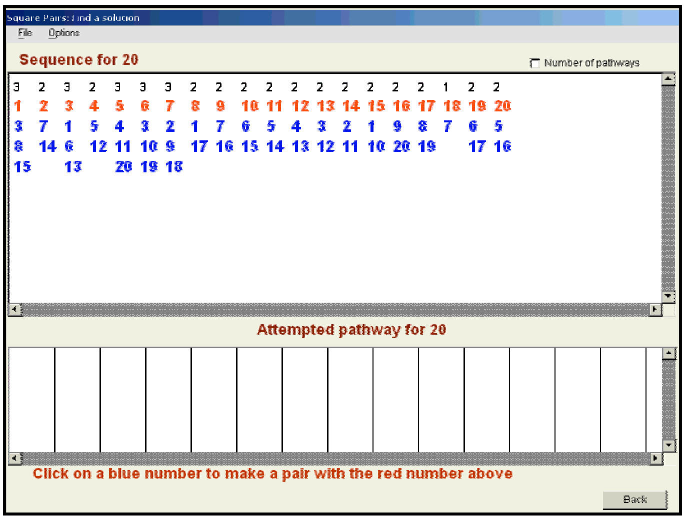
The people holding their number cards are in red. The blue numbers below each one are the only other numbers with which they can square pair. The layout makes the 'must do' square pairs obvious. No question about (2, 7) and (4, 5). Further, (6, 3) and (8, 1) show why the apparent pairing choices for 1 and 3 aren't really choices at all. As students follow the instruction in red, their attempt (which might not be as just described) is recorded and blue numbers turn pink as they are used.



Dipping into the strategy toolbox to explain lists that don't work can be simple for some lists, eg:



but for others is quite sophisticated, eg:



In this list (18, 7) is a must do pair because if 7 is used with anything else there will be no pair for 18. Making this choice creates the must do choices (2, 14) and (9, 16) because 7 has just been used. Making these must do choices creates an impossible must do choice for the next move because 5 must then pair with 11 and it must also pair with 20. In the language of the strategies, we have found an exception. In the more formal language of the mathematician, we have followed valid reasoning and come to an absurdity, so the original proposition (the implied one that 1 - 20 could be square paired) must be false. This type of reasoning is called Reductio ad Absurdum - reduction to an absurdity.

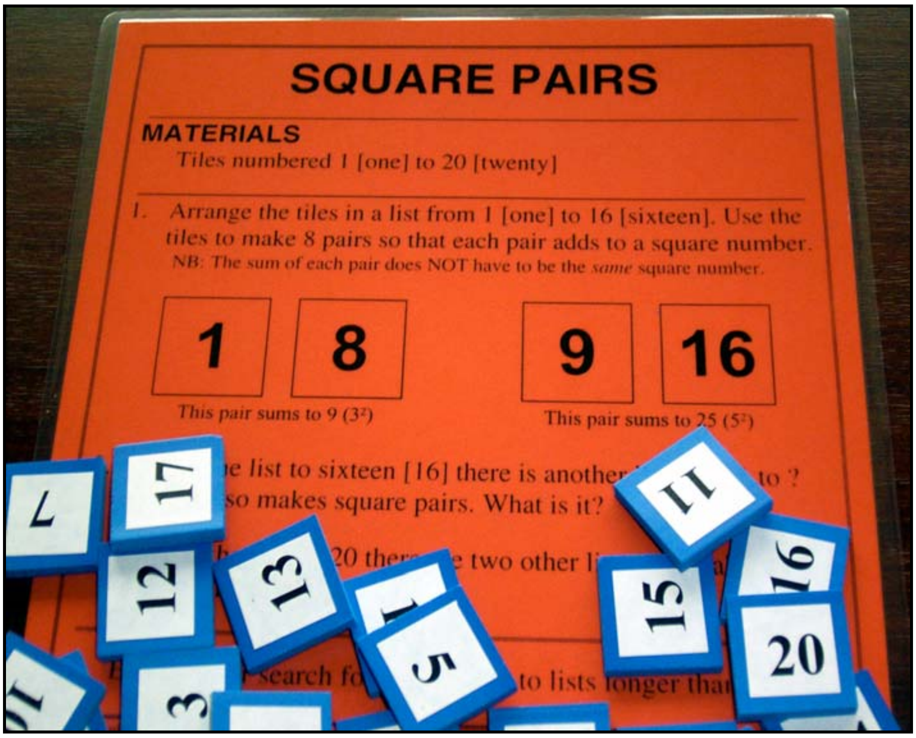
And that language brings us back to Anderson & Walker.

> *...there are aspects of this problem and its solution ... which could form the basis for an investigative project for pupils, involving experimentation (either on paper or through simple computer programs), conjecture and, more importantly, an opportunity to try to construct proofs.*

Investigation, experimentation by hand and computer, conjecture and an opportunity to construct proof have all been included. Any teacher following this lesson outline is supporting students to learn to work like a mathematician and can show why, where and how by reference to the Working Mathematically page. In the process there will be point of need teaching and skill practice to do with summing numbers and square numbers, but you will notice our authors didn't suggest the challenge *for the purpose of* this lower level stuff.

There is more to the problem of course - there always is - such as realising that finding seven cases that don't square pair doesn't prove these are the *only* ones with that property. That's what our modern day mathematicians showed; that these seven cases are unique among the infinity of possible even numbers of party guests. However, this is not the time to pursue the lesson further. Rather let's look at the curriculum balance model again. We have been practising some skills in a best practice classroom which is modelling how a mathematician works. There have been opportunities within that for students to work on their own or in small groups. But can we use the problem in a different way, a way that encourages students to apply the model for themselves without the safety net of the whole class working on the same problem.

That's the intent of Square Pairs as a hands-on task from the Mathematics Task Centre:



Now the problem comes with enough material and information for two students to 'start swimming'. In the same way as a book in the library can be a personal adventure for a child, or converted to a whole class literary adventure by a teacher, these tasks provide opportunity to invite students to swim in the great ocean of mathematical adventure or opportunity for the teacher to take the whole class swimming in the same ocean.

And yes there is a very big ocean of possibilities. The Task Centre collection alone offers 239 of them, all as rich as this, and Maths300 offers more.

There is no need for maths to be taught the way it always has been.

References

Anderson, Johnston & Walker, Andy: *Partitions into Square-Pairs*, Mathematical Gazette, Issue 496, 1999. Department of Mathematics, University of Nottingham, Mathematical Association, UK.

HREF1: http://www.blackdouglas.com.au/taskcentre, Mathematics Task Centre, Black Douglas Professional Education Services, Australia

HREF 2: http://www.curriculum.edu.au/maths300, Maths300, Curriculum Corporation, Australia

Working Mathematically

First give me an interesting problem.

When mathematicians become interested in a problem they:

- Play with the problem to collect & organise data about it.
- Discuss & record notes and diagrams.
- Seek & see patterns or connections in the organised data.
- Make & test hypotheses based on the patterns or connections.
- Look in their strategy toolbox for problem solving strategies which could help.
- Look in their skill toolbox for mathematical skills which could help.
- Check their answer and think about what else they can learn from it.
- Publish their results.

Questions which help mathematicians learn more are:

- Can I check this another way?
- What happens if …?
- How many solutions are there?
- How will I know when I have found them all?

When mathematicians have a problem they:

- Read & understand the problem.
- Plan a strategy to start the problem.
- Carry out their plan.
- Check the result.

A mathematician's strategy toolbox includes:

- Do I know a similar problem?
- Guess, check and improve
- Try a simpler problem
- Write an equation
- Make a list or table
- Work backwards
- Break the problem into smaller parts
- Act it out
- Draw a picture or graph
- Make a model
- Look for a pattern
- Try all possibilities
- Seek an exception
- ...

If one way doesn't work I just start again another way.