Mathematics Education is not an Enigma

Doug. Williams
Project Manager, Mathematics Centre
http://www.mathematicscentre.com ... doug@blackdouglas.com.au

This paper is an edited version of the opening address given at the Association of Teachers of Mathematics (ATM) Easter Conference 2012, Swansea University, Wales. The theme of the conference was 'Enigmas'. As they entered the theatre for the address each person was given a paper cup in preparation for the pair activity on Page 6. You will need two paper cups.

Setting the Scene

I am a storyteller in mathematics education. It is my job to collect and retell stories of success from classrooms. It is the quantity and quality of these stories which leads me to the belief that there is nothing mysterious, puzzling or difficult to understand about mathematics education. We know how to do it and do it well. Therefore mathematics education is not an enigma.

The theme which weaves these separate stories into an integrated anthology is choosing the core of the curriculum to be learning to work like a mathematician (Working Mathematically) and choosing teaching craft that fascinates, captivates and absorbs learners. In such classrooms the teacher realises that they are *not* teaching mathematics and the children realise that they *are* involved in the process of working like a mathematician in a happy, healthy, cheerful, productive, inspiring classroom.

Mathematics content - the stuff you see listed in systemic curriculum documents - offers very little in and of itself. The amount listed cannot be justified on the basis of 'you need it to pass an exam', or 'it will help you to get a job one day', or 'you might need this when you grow up'. Worse, lists of content set learners up to fail. A content-driven view of mathematics learning ensures everyone succeeds ... to the first level of content that they 'don't get'... and often, having not 'got that', they decide they can't do maths. Some of these learners later find themselves teaching mathematics with no vision of a mathematics classroom to inspire them other than the content-driven ones they survived ... or escaped from.

Content-driven curriculum sets teachers up to fail too. It encourages us to think about teaching stuff rather than people; whole human people with an integrity and a self-image formed by past experiences and within which they operate.

In 1969, following my degree, I studied a Diploma in Education course at Monash University and, through the wisdom and teaching experience of lecturers and tutors across the course, in particular David Davidson, my Maths Methods tutor, I began my teaching career believing that it *was not* my job to teach mathematics. Rather, it *was* my job to use my mathematics teaching to help learners feel better about themselves - to build their self-esteem. This is an objective I have failed to achieve with many of my students through my years of full time teaching in both primary and secondary schools from 1970-1991, but it is one from which I have never wavered.

I was fortunate that my attitude was fashioned in this way before being qualified. However it took many more years, many experiences and the stories told by many colleagues before I recognised that there is an alternative framework to traditional mathematics teaching and that when teachers choose to change their attitude to make this framework the core of their teaching, students learn more mathematics content more quickly and securely and feel successful doing so.

The turning point for myself and others was the clear, concise statement of the work of a mathematician developed by the Mathematics Task Centre Project from the statements of professional mathematicians. This Working Mathematically framework is appended and may be freely reproduced and distributed. It helped us move from problem solving being added to the curriculum as a topic, to working like a mathematician being the essence, the core, the raison d' être, that gives all mathematics content its meaning.

When asked about their work, mathematicians began their response with "First give me an interesting problem."

So, a Working Mathematically curriculum would *begin* with problems - not add them on at the end of a chapter if there was time, or reserve them for one period per week - and our teaching craft would be called on to interest learners in the problems. That's very different to a content-driven curriculum. And all it takes to achieve the shift is a change of teacher attitude.

Working Mathematically Classrooms

Soon I will introduce you to teachers who, regardless of the official curriculum constraints under which they work, changed their attitude to mathematics teaching in the way I have described. But first, let's look for features of a Working Mathematically classroom likely to encourage learning. What features likely to encourage learning do you see in these photos?

Photo 1

Photo 2

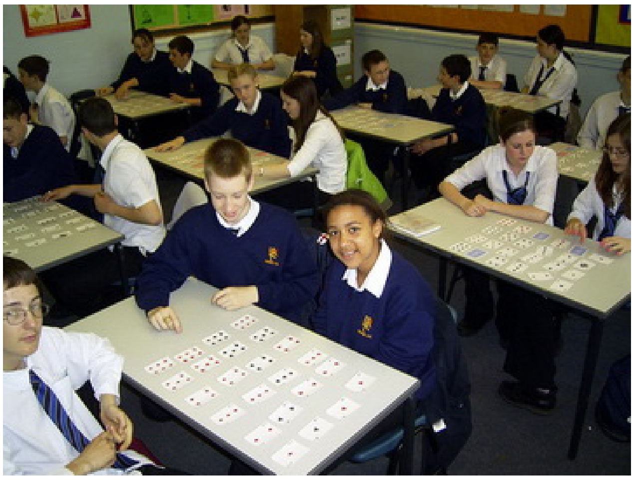


Photo 3

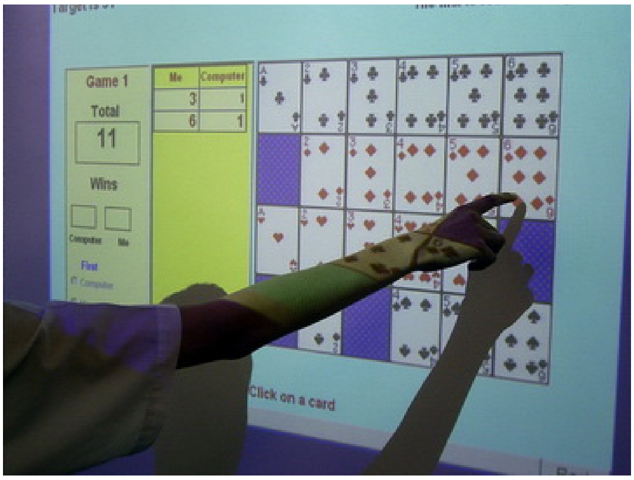


Photo 4

Photo 5

Photo 6

Photo 7

Photo 8

Perhaps you would list teaching craft such as...

- working outside
- whole body involvement
- estimation
- student teaching student
- seeing rather than hearing maths
- concrete
- tactile
- recording
- discussion (teacher/student)
- questioning rather than telling
- recording
- mathematical conversation
- choice of tasks
- common task with different outcomes
- game
- competition
- connection with home
- software
- student control

each one a deliberate pedagogical decision taken by the teacher.

Now look back through the photos and add this geographical information to your considerations:
- Photo 1: Trädgårdsstadsskolan, Tullinge, nr. Stockholm, Sweden
- Photo 2: Silvan Primary School, nr. Melbourne, Australia
- Photos 3, 4: Thorne Grammar, nr. Doncaster, UK
- Photos 5, 6: Wade High School, nr. Griffith, Australia
- Photos 7, 8: Thorne Grammar, nr. Doncaster, UK

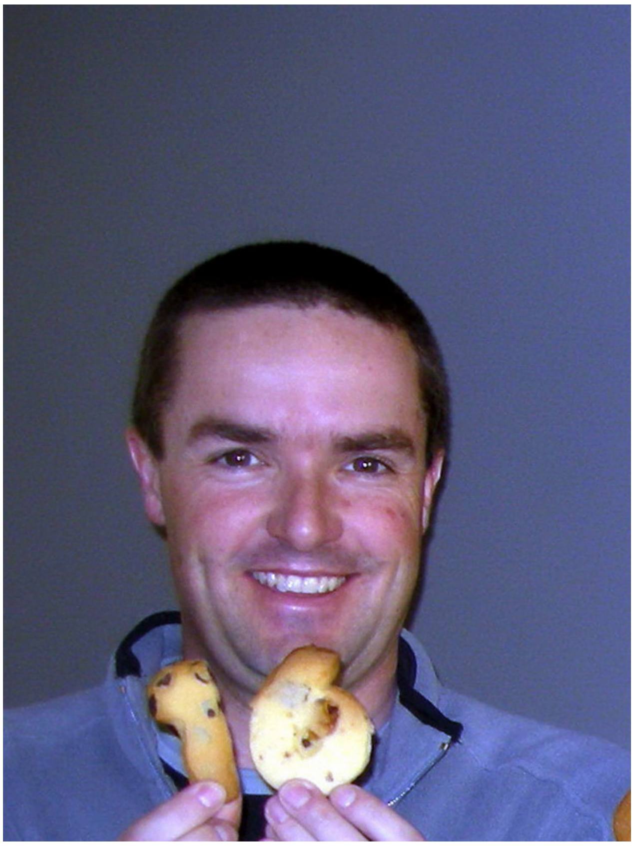
If you also knew that these distant classrooms are using the same resources to plan their curriculum, then you might realise the possibility of international exchange between students as they struggle with a common problem. Try achieving that with a text book.

Now suppose I ask: "If government, the media or Hollywood photographed mathematics lessons what would they be likely to show?". Your answer is more likely to be something like:
- students in rows
- heads down
- teacher at board
- board full of symbols

This is a preconceived idea of learning mathematics that belongs in the Industrial Revolution where it originated. It is irrelevant to the present and future of current students and it is only teachers who change their attitude who can change that image?

Working Mathematically Teachers

Meet ...



Damian Howison ... and his staff of Mackillop College, Swan Hill, Australia.

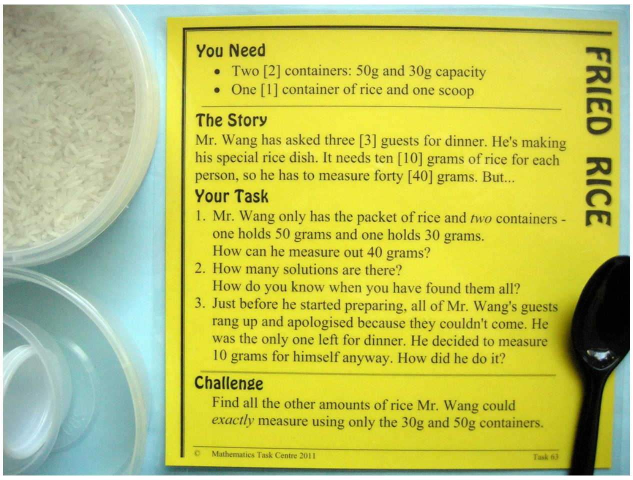
The photos were taken during a professional development session on National Mathematics Day 2008. They had been working together on this problem...

...and were celebrating the fact that they had found several solutions, by eating them.

Since then they have used Task 35, Crosses (HREF1), each year during the orientation program for new Year 7 students. I'm not sure whether students also get to eat their solutions!

Damian and his staff have just entered their seventh year of building their curriculum around working like a mathematician.

I want to try to illustrate how their thinking operates so you can make your own judgement about how it relates to what is often described as traditional mathematics teaching - or should I use the term 'default mathematics teaching'.



FRIED RICE

You Need

- Two [2] containers: 50g and 30g capacity
- One [1] container of rice and one scoop

The Story

Mr. Wang has asked three [3] guests for dinner. He's making his special rice dish. It needs ten [10] grams of rice for each person, so he has to measure forty [40] grams. But...

Your Task

1. Mr. Wang only has the packet of rice and *two* containers - one holds 50 grams and one holds 30 grams. How can he measure out 40 grams?
2. How many solutions are there? How do you know when you have found them all?
3. Just before he started preparing, all of Mr. Wang's guests rang up and apologised because they couldn't come. He was the only one left for dinner. He decided to measure 10 grams for himself anyway. How did he do it?

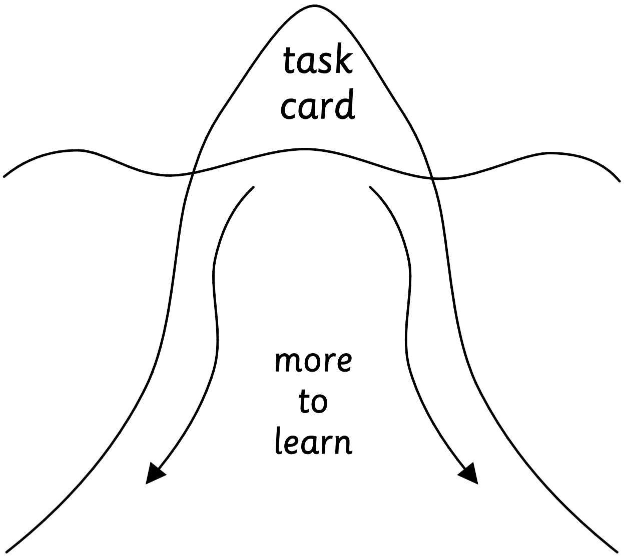
Challenge

Find all the other amounts of rice Mr. Wang could *exactly* measure using only the 30g and 50g containers.

© Mathematics Task Centre 2011 Task 63

Mackillop has a large collection of tasks and this is one of them (HREF2). Just read the story shell and Question 1 for now.

A task like this has three lives. In this form it is an invitation to a pair of students to work like a mathematician. The problem of course could be in a text book, in fact it can be found in Polya's classic book *How To Solve It?* (in the last section of the book in an exposition of working backwards), but a defining feature of a task, as opposed to a book exercise, is that the presentation of the problem includes materials students can use to investigate it.



A good task is the tip of an iceberg.

This invitation can be extended into what becomes a second life by writing an Investigation Guide to extend students beyond the tip of the iceberg. The team has written a guide for this task, but on this occasion they wanted to use the task in its third life, as a whole class investigation, so they converted it to...

"ok kids…today I have a little challenge for you…."

Jack and Jill went up the hill

To collect exactly 4 litres of water.

Jack had a 5 litre bucket

And Jill had a 3 litre bucket.

But they managed the task. How?

Source: Damian Howison, Mackillop College, Swan Hill, Victoria

Which allowed Damian and the team to make this teaching craft decision...

...because they believe that more important than the numbers in the problem is the opportunity to play with those numbers in context.

Now you are invited to solve Jack and Jill's problem. Perhaps you won't want to find two containers the right size and some water, but at least find two paper cups and pretend. It does make a difference when attempting to solve the problem. It also makes a difference if you try to solve Jack and Jill's problem in discussion with someone else.

---------------- *pause here and try the problem* ----------------

When a mathematician solves a problem they have to explain the solution to colleagues. Imagine you need to send a fax to a friend to explain your solution. A fax allows you to use drawings as well as text. Try it...

- Does the process of preparing the fax cause you to rethink the problem and consolidate your learning?

- Does rethinking the problem generate new thoughts?

Working Mathematically Questions

One strategy a mathematician could use to explain the solution would be as a table.

	3	5	
	0	0	
Fill --->	3	0	
	0	3	
Fill --->	3	3	
	1	5	
	1	0	---> Empty
	0	1	
Fill --->	3	1	
	0	4	

To solve the problem, the 3 litre bucket has been filled 3 times and the 5 litre bucket emptied once.

Now a mathematician might ask: "Can I check the solution another way?" and perhaps try beginning by filling the 5 first. Try it ... it actually generates another solution ... and try to record the new solution in a table too.

Given two solutions a mathematician must ask: "How many solutions are there?" and "How do I know when I have found them all?" In this case there are only two ways to begin, the process is mechanical after that, so there must only be two solutions. The second solution involves filling the 5 litre bucket twice and emptying the 3 litre bucket twice.

So the two solutions could be represented as:

$$(3 \times 3) - 5 = 4 \text{ and } (2 \times 5) - (2 \times 3) = 4$$

Already an iceberg is beginning to appear for this problem and we can still ask: "What happens if we change the buckets?"

"ok kids…today I have another little challenge for you…."

Jack and Jill went up the hill

To collect exactly 6 litres of water.

Jack had a 9 litre bucket

And Jill had a 4 litre bucket.

But they managed the task. How?

Source: Damian Howison, Mackillop College, Swan Hill, Victoria

This is the form of the problem that appears in Polya's book. But the text makes no mention of any other questions that could be asked, such as:

- Can we choose any two buckets and expect to fill to an amount between the two measures?

- What happens if we continue the filling and emptying process beyond the arbitrary amount Jack and Jill were asked to collect? After all, from the table above, Jack and Jill could also have taken home 3 litres (by stopping at [0, 3]) or 1 litre [0, 1]. Is it not possible that if they continued they could take home 2 litres as well? Then they would have been able to use the two buckets to bring home any whole number amount of water from 0 litres to 5 litres. Wow!
- What happens if we draw a diagram to represent the order in which the 5 litre bucket is filled and the 3 litre bucket is empty? That is, a diagram that shows the sequence of all the possible take home amounts.

Look back at Mr Wang's rice task with eyes that are now more experienced and you will see some of this adventure hinted at in the other questions on the card.

All Mackillop's work - which could easily be up to a week of mathematics lessons around one investigation - is driven by best practice teaching craft in a core curriculum built around learning to work like a mathematician.

First give me an interesting problem

Mathematicians really do mean this. They don't say first let me count, or first give me algebra, or first let me apply a formula. They say first give me an interesting problem ... and it can be anything.

In order to change our attitude to mathematics teaching, perhaps first give me an interesting problem could be our mantra. For example: "I have to teach fractions tomorrow, so first I look for a problem likely to generate fractions and then I review my teaching craft options to find a way to fascinate, captivate and absorb my learners."

However, we went further when discussing with our mathematician friends and developed the appended list of their key 'ways of knowing'. This is simple, yet subtle, description has brought success to hundreds of classrooms from K to 12. It provides a higher order curriculum focus and a language for exploring it. Schools having most success publish and use this language consistently across the school. And, perhaps because it makes sense, students don't ask "Why do we have to do this?"

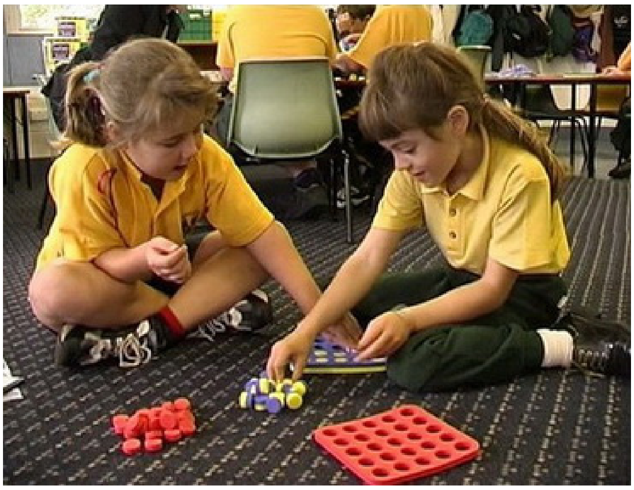
> *I used Birth Month Paradox* (HREF3) *with two Year 10 extension maths classes and one Year 11 VCAL class. The odd thing with the VCAL class is that I keep expecting them to say "What's the point of doing these types of questions?" ...but they never do.*
> Email: Ian Pegram, Dandenong High School, 24th July 2007

So, at all year levels, a Working Mathematically curriculum is built around problems - problems with unknown answers, not recipe exercises with expected answers - and the teaching craft that fascinates, captivates and absorbs learners.

For example, in Ian Pegram's class, who gives a 'tinkers' about how many people you need in a room to have a good chance of having two with the same birthday? Pretty much no one. You will never need to know that fact. But on the other hand, who gives a tinkers about the Birth Month Paradox investigation? Well, for a start, and apparently unexpectedly, all of Ian's Year 10 and 11 classes!

First give me an interesting problem works in primary school too. I will let Jacki Healey, Margate Primary School, tell this story of Poly Plug Values as it unfolded in her class. It is recorded in the Calculating Changes section of Mathematics Centre.

I was working with Place Value with my Grade 2 class and I wanted to do something to extend their understanding. I decided to use Poly Plugs. The first time I introduced the Poly Plugs I told them it would be interesting to use them to represent numbers.
How could they do this?
Initially I used low numbers, eg: 14, 17, etc. I discovered that given low numbers, most children would assign each Poly Plug a value of 1, ie: 1 to 1 correspondence.
The challenge lay in giving them larger numbers to represent, eg: 96, 43, 85, 74. Large numbers and odd numbers were harder and required more ingenuity to represent. At the end of the first 1 hour session, some children had considered giving, for example, a red plug the value of 1, a yellow 2, a blue 5 or 10.



The yellow is worth 2.
We are counting by twos...

...all the way to 48.
But what will this last one be??

Over the next 3 weeks I repeated the exercise, giving each group the opportunity to swap partners and share ideas. It was tremendously exciting to watch as the children's confidence and ingenuity increased each week.
They were also asked to record their ideas, so that they could share with the whole group. By the end of the month the children's ability to 'skip count' in 2s, 3s, 4s, 5s, 6s and 10s was amazing. Some children even assigned value to the empty places in the plug board. They also thoroughly enjoyed the whole process.

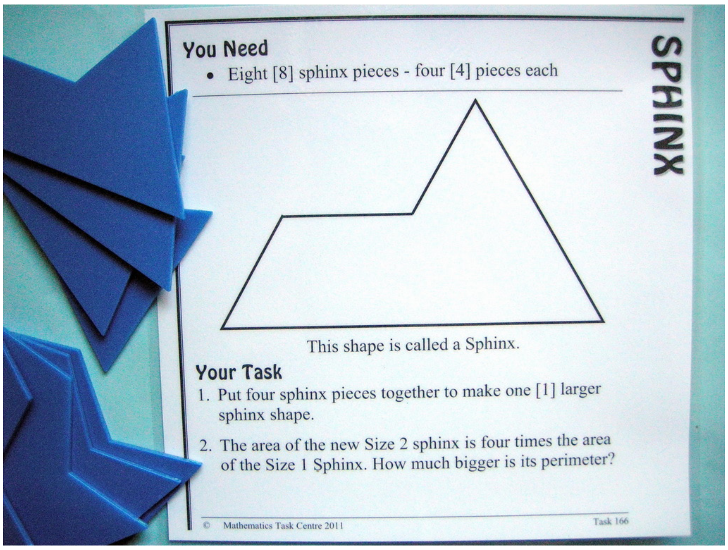
Jacki is using a teaching method here that has been devised by teachers in the Calculating Changes network (HREF4). It's called Threading. The same rich task with a familiar structure is used for small amounts of time, often, over several weeks. The learner's intellectual energy is invested in the mathematics of the activity, rather than the structure of the activity, and developing learning can be easily tracked. Jacki's story continues:

One day I suggested that they had worked so hard they deserved a break. They could play on the brand new playground equipment, then come back to finish their Poly Plug Values. They unanimously refused and chose to complete the task first!

What is of primary importance here? The mathematics content or the classroom environment Jacki created and the way the children felt about themselves as learners?

All it takes to develop a classroom with a Working Mathematically core curriculum is a teacher - you - who adopts a change of attitude; and that adoption is possible independent of any externally impressed curriculum force.

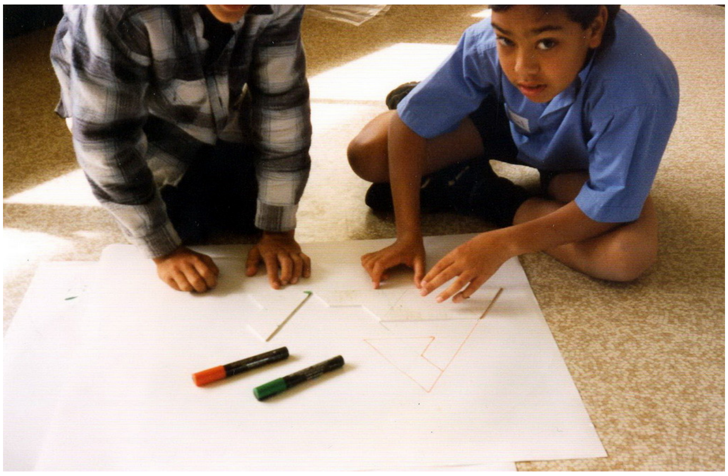
Sphinx



This task is the most extensively documented of any (HREF5, 6, 7). However some may not be familiar with it, so a quick background. In the original form of the task, students were given four wooden pieces shaped as shown and challenged to arrange them into a larger 'Sphinx' shape.

As it stands the task is a spatial challenge of moderate difficulty. At a teaching day in 1996 which initiated the project to trial the use of tasks in an urban Indigenous situation, two Year 5 Aboriginal and Torres Strait Islander students, Tyler and Michael from Norris Road School, accepted the challenge. They solved it quickly, but then were challenged to investigate further into the iceberg.

Their solution showed that the base, height and side lengths of the new Sphinx were twice as long as the corresponding lengths in a single Sphinx shape. The boys' teacher pointed this out and asked them to relate this '2 times' information to the fact that the new Sphinx was made of four original ones. The discussion led to recognising a growth pattern Size 1, Size 2, Size 4... and led to the boys asking: *What happened to Size 3?*



Their teacher did not know the answer and, at this stage, did not even know if was possible to solve this extension of the problem. The boys were invited to take on this new challenge, made all the more difficult by the fact that only four wooden shapes were available. They found a large sheet of paper and set about tracing. This can be a frustrating approach because, once traced, the drawing can't be picked up and rotated or reflected like the wooden piece can. But they persevered. In fact they stuck at the task for about 45 minutes and were still trying when time ran out.

This teaching day was held at a central location and included students from several schools. Michael and Tyler had to return to their own school. They chose to take their incomplete work with them. Weeks later their solution to this quite difficult task was delivered to the project organisers.

Significantly the boys had claimed their solution by superimposing a drawing of the Aboriginal flag on their nine piece solution.

What is of primary importance here? The fact that 4 funny shapes fit together to form the same funny shape? The fact that the growth pattern leads to work in topics of patterns and powers, area and perimeter, prime and composite numbers, algebraic generalisation, symbolic representation and more. Or how those boys felt about themselves?

Adventure of the Sphinx

One part of the Sphinx story is told in a video (HREF10) one class made to publish the results of their investigation, just as a mathematician would.

Dani and her class, with the support of teacher Pamela McGifford, made this video in 1999. In March 2009 Dani, whom I have never met, wrote to me out of the blue asking if I knew where she could get a copy.

> *I used to be a student at Cressy District High School where we studied the Sphinx problem in depth with our teacher Pam McGifford. As I'm sure you are aware, we recorded a video detailing our discoveries. I am currently in the process of trying to track down a copy of this video (I don't think I ever actually watched it!), and was wondering if you would be able to help me with this.*

That started a conversation which included:

> *How has the Sphinx influenced my life to present? Well I will admit I have never been a mathematically minded person. Far from one in fact! However at the time it was a fantastic learning experience. I have always been a good student and achieved high marks but mathematics was always a foreign language to me so I didn't have any interest in that area. Studying the Sphinx problem sparked my interest, and got me actively involved in something I could piece together and evaluate, rather than just work out on paper. Because I was interested by it all, I actually learnt a lot without realising it, and had a strong foundation to relate other problems to in the future. So in that sense, I suppose the Sphinx has helped me no end. I am still terrible at maths but I have some good practical knowledge arising from the Sphinx problem that I can draw upon if I need to.*

What is of primary importance here? The fact that 4 funny shapes fit together to form the same funny shape? The fact that the growth pattern leads to work in topics of patterns and powers, area and perimeter, prime and composite numbers, algebraic generalisation, symbolic representation and more. Or how Dani and her classmates felt about themselves?

Dani continued the email conversation with:

> *We had Pam McGifford for Maths and John Bradbury for science, although I think he used to fill in for Pam from time to time and take our maths class. I'm not sure what to say about them, apart from how fantastic they are! They are both excellent teachers with that essential (well I think it is anyway) ability to relate to the students on a personal level whilst still maintaining the student-teacher order and respect. I only hope that when I complete my studies one day I can be half as good as them.*

Pam and John were both involved in the INISSS professional development research with 40-60 secondary teachers (HREF8, 9). INISSS stands for Improving Numeracy for Indigenous Secondary School Students and it provides research evidence which confirms the stories I have been telling. Both a descriptive and a more formal outline of the research can be found in the References. In summary, this was a four year research program based around professional development intended to shift teachers' attitude towards learning to work like a mathematician. The outcomes were:

- Students had better problem solving skills...
- Students had better content skills...
- Students had better literacy skills...

...than equivalent cohorts whose teachers had not been involved in the professional development. More importantly the program was stimulated by a desire to improve learning outcomes for Indigenous students. Through helping teachers shift their vision of mathematics education, by Year 10, the Indigenous student group, who were significantly 'behind' non-indigenous students when entering secondary school in Year 7, had essentially reached parity with their classmates. All previous evidence was of the gap widening over these years, not narrowing - and certainly not disappearing.

So, is mathematics of primary importance? No.

- Why we teach is of primary importance.
- Learners are of primary importance.
- How we teach them is of primary importance.

The mathematics itself doesn't really matter - but learning to work like a mathematician in a best practice classroom does matter. In this context, many, many, many teachers have found that there is nothing mysterious, puzzling or difficult to understand about mathematics education. We know how to do it and do it well. Therefore mathematics education is not an enigma.

References

HREF1: http://www.mathematicscentre.com/taskcentre/035cross.htm, Mathematics Centre, Black Douglas Professional Education Services, Australia (Crosses task)
HREF2: http://www.mathematicscentre.com/taskcentre/063fried.htm, Mathematics Centre, Black Douglas Professional Education Services, Australia (Fried Rice task)
Polya, George (1957). *How to Solve It*. Garden City, NY: Doubleday
HREF3: http://www.maths300.com, Australian Association of Mathematics Teachers (Birth Month Paradox is a members' lesson)
HREF4: http://www.mathematicscentre.com/calchange, Mathematics Centre, Black Douglas Professional Education Services, Australia (Calculating Changes Network - Poly Plug Values is from the Members section)
HREF5: http://www.mathematicscentre.com/taskcentre/166sfinx.htm, Mathematics Centre, Black Douglas Professional Education Services, Australia (Sphinx task)
HREF6: http://www.mathematicscentre.com/taskcentre/sphinx.htm, Mathematics Centre, Black Douglas Professional Education Services, Australia (album of Sphinx information)
HREF7: http://www.mathematicscentre.com/taskcentre/logo.htm, Mathematics Centre, Black Douglas Professional Education Services, Australia (Michael & Tyler's story)
HREF8: http://www.mathematicscentre.com/taskcentre/do.htm#inisss, Mathematics Centre, Black Douglas Professional Education Services, Australia (INISSS research)
HREF9: http://www.mathematicscentre.com/taskcentre/cressy01.htm, Mathematics Centre, Black Douglas Professional Education Services, Australia (evidence from Cressy District High School)

Additional References (2019)

An earlier version of this paper was published in two parts in *Mathematics Teaching*, Association of Teachers of Mathematics, UK, Issue 230, September 2012 and Issue 231, November 2012. Differences between the two issues are few and include the update to the Maths300 reference above and the following reference to the Sphinx video which is now available.

HREF10: http://mathematicscentre.com/mathematicscentre/cubetube.htm#sphinx, Mathematics Centre, Black Douglas Professional Education Services, Australia (gateway to the video stored on Cube Tube, the Mathematics Centre YouTube channel)

Working Mathematically

Learning to Work like a Mathematician

First give me an interesting problem.

When mathematicians become interested in a problem they:

- Play with the problem to collect & organise data about it.
- Discuss & record notes and diagrams.
- Seek & see patterns or connections in the organised data.
- Make & test hypotheses based on the patterns or connections.
- Look in their strategy toolbox for problem solving strategies which could help.
- Look in their skill toolbox for mathematical skills which could help.
- Check their answer and think about what else they can learn from it.
- Publish their results.

Questions which help mathematicians learn more are:

- Can I check this another way?
- What happens if …?
- How many solutions are there?
- How will I know when I have found them all?

When mathematicians have a problem they:

- Read & understand the problem.
- Plan a strategy to start the problem.
- Carry out their plan.
- Check the result.

A mathematician's strategy toolbox includes:

- Do I know a similar problem?
- Guess, check and improve
- Try a simpler problem
- Write an equation
- Make a list or table
- Work backwards
- Break the problem into smaller parts
- Act it out
- Draw a picture or graph
- Make a model
- Look for a pattern
- Try all possibilities
- Seek an exception
- ...

If one way doesn't work I just start again another way.