Jumping Kangaroos

Investigation Guide: Discussion

2. The numbers in this table represents those for the kangaroos to swap places. (To just pass each other would require one less move.)

No on each side	1	2	3	4	5	6	7
No. of moves	3	8	15	24	35	48	63

4.

No on each side	10	20	100	...	n
No. of moves	120	440	10200	...	$n(n + 2)$

Students usually see a rule for predicting the number of moves which is to multiply the number of pieces by 'a number bigger by 2',
ie: n x $(n + 2)$. But seeing the logic of why this might make sense is very challenging. The section below, which might be more for senior students, attempts to do this.

5.

No on each side	9	11	31	50	108
No. of moves	99	143	1023	2600	11,880

No on each side	76	999
No. of moves	5928	999,999

- Patterns of moves
  Once students realise how important it is to set up an alternating pattern between the two sides, the pattern of these moves can be recorded. For example, if the two sides are called A and B pieces, with two on each side the move sequence is:
  A B B A A B B A

  Writing these as numbers of moves from one side and then the other, the sequence is: 1 2 2 2 1 for a total of 8 moves.

  The table for these patterns of moves is:

Number of pieces	Pattern of moves	Total
1	1 1 1	3
2	1 2 2 2 1	8
3	1 2 3 3 3 2 1	15
4	1 2 3 4 4 4 3 2 1	24
5	1 2 3 4 5 5 5 4 3 2 1	35

The pattern of increases can now be seen. For example, the pattern for 10 on each side would be:

1 2 ... 8 9 10 10 10 9 8 ... 2 1 for a total of 120.

If there were now 11 on each side the pattern would be:
1 2 ... 9 10 11 11 11 10 9 .... 2 1 for a total of 143.
The increase is the extra 3 lots of 11 in the middle and one less 10.

Generally for *n* pieces the increase will be 3 lots of *n* less one lot of (n - 1), and since $3n - (n - 1) = 2n + 1$, the increase is always $2n + 1$ over the previous answer.

- A proof for $n \times (n + 2)$
The formula $n(n + 2)$ is an elegant and concise summary of the pattern but it does not show the logic behind the puzzle, which the series of moves can show. For *n* pieces on each side the pattern of moves is:
<u>1 2 3 ... (*n* - 1) *n*</u> *n* <u>*n* (*n* - 1) ... 3 2 1</u>

That is, two arithmetic series from 1 to *n* plus an extra *n* (in the middle).

Each of the series 1 2 3 (*n* - 1) *n* has a sum of $\frac{n}{2}(n+1)$ - from the formula for the Sum of an Arithmetic Series. A similar pattern appears in the Staircase task.

Hence the total is:
$1 + 2 + 3 ... + (n - 1) + n \quad + n + \quad n + (n - 1) + ... 3 + 2 + 1$
$= \frac{n}{2}(n+1) \quad + n + \quad \frac{n}{2}(n+1)$
$= 2 \times \frac{n}{2}(n+1) + n$
$= n(n + 1) + n$
$= n^2 + 2$
$= n(n + 2)$

Well that's an analysis of the problem in algebraic symbols, but can we check it another way? There are two terms in $n(n + 2)$ and two types of move in the problem:

- jumps over one kangaroo
- hops into an empty space

If algebra makes sense there must be some connection between the moves in the problem and the rule describing the answer. Perhaps we should look at a certain kangaroo, the number of times it has to jump others and the number of times it has to hop into an empty space.

See what you can find.